FINITE-DIMENSIONALITY OF THE SPACE OF LIPSCHITZ HARMONIC FUNCTIONS ON A VERTEX-TRANSITIVE GRAPH: A SHORT PROOF

SEAN EBERHARD AND FREDDIE MANNERS

ABSTRACT. Kleiner gave a new proof of Gromov's theorem on groups of polynomial growth by demonstrating directly that the space of Lipschitz harmonic functions on a finitely generated group of polynomial growth is both nontrivial and finite-dimensional. In this paper, adapting some ideas of Ozawa, we present a new and shorter proof of Kleiner's theorem. Notably, our proof produces a dimension bound for the space of Lipschitz harmonic functions that is linear in the degree of growth, while Kleiner's original argument gives only an exponential bound. We also briefly discuss potential implications of this method in establishing new lower bounds for the growth of non-virtually-nilpotent groups.

1. INTRODUCTION

The celebrated theorem of Gromov [G2] states that finitely generated groups of polynomial growth are virtually nilpotent. This theorem has since influenced many areas of geometric group theory and related fields. While Gromov relied on the solution to Hilbert's fifth problem, Kleiner [K] (building on ideas of Colding–Minicozzi) later gave a new proof by examining the space of Lipschitz harmonic functions on groups of polynomial growth, showing that this space is always nontrivial and finite-dimensional. It follows that any infinite finitely generated group G of polynomial growth admits a finite-dimensional linear representation $G \to \mathrm{GL}_n(\mathbf{C})$ with infinite image. The proof of Gromov's theorem is completed by appealing to the Tits alternative and applying induction on the degree of polynomial growth.

More recently, Ozawa [O] gave a functional-analytic proof of Gromov's theorem based on reduced cohomology, Shalom's property $\mathcal{H}_{\mathrm{FD}}$, and entropy arguments.

In this paper we give a new short proof of Kleiner's theorem on the finite-dimensionality of the space of Lipschitz harmonic functions. Our method borrows some ideas from the work of Ozawa but applies them directly in the concrete context of Lipschitz harmonic functions. We thus obtain a comparatively short proof of Gromov's theorem.

We work in the setting of an arbitrary vertex-transitive graph, as the proof is no more difficult in this (superficially) more general context, and indeed some aspects are clarified. Fix a connected, locally finite, vertex-transitive graph Γ. The motivating case

SE is supported by the Royal Society through a University Research Fellowship (URF\R1\221185). FM is supported by a Sloan Fellowship.

is $\Gamma = \mathrm{Cay}(G, S)$ where G is a finitely generated group, $S = S^{-1}$ is a symmetric finite generating set not containing the identity, and $\Gamma = (G, E)$ where

$$E = \{\{g, gs\} \colon g \in G,\ s \in S\},$$

so that G acts vertex-transitively on Γ on the left by graph automorphisms. The reader will lose nothing by working exclusively in this case if preferred. Write r for the common degree of every vertex $v \in V(\Gamma)$ (so, $r = |S|$ in the case of a Cayley graph $\Gamma = \mathrm{Cay}(G, S)$). In general, we write $v \sim w$ to mean that $vw \in E(\Gamma)$ and we write $N_\Gamma(v)$ for the set of vertices adjacent to v.

A function $f : V(\Gamma) \to \mathbf{C}$ is called *Lipschitz* if

$$\|f\|_{\mathrm{Lip}} := \sup_{v \sim w} |f(v) - f(w)| < \infty$$

and *harmonic* if

$$f(v) = \frac{1}{r} \sum_{w \sim v} f(w) \qquad (v \in V(\Gamma)).$$

Let $\mathcal{H}_L(\Gamma)$ denote the space of Lipschitz harmonic functions on Γ.

Finally, let $B_\Gamma(v, n)$ denote the ball of radius n around a vertex $v \in V(\Gamma)$ and write $b(n) := |B_\Gamma(v, n)|$. Note that, by vertex transitivity, $b(n)$ is indeed independent of v, as the notation suggests. We say that Γ has *weakly polynomial growth of degree d* if

$$d = \liminf_{n \to \infty} (\log b(n) / \log n) < \infty.$$

The following is our main result.

Theorem 1.1. *Suppose Γ is a connected, r-regular, vertex-transitive graph with weakly polynomial growth of degree d. Then*

$$\dim \mathcal{H}_L(\Gamma) \le 4r^2 d + 1.$$

Remarks.

(1) In the other direction, we have $\dim \mathcal{H}_L(\Gamma) > 1$ whenever Γ is infinite. For this easier result, see the original argument of Kleiner [K, §4], Lee–Peres [LP], or Shalom–Tao [ST, §6]. The last was recently adapted to the vertex-transitive setting in [ABGK].

(2) Kleiner's proof that $\dim \mathcal{H}_L(\Gamma) < \infty$ applies more generally when the space of Lipschitz harmonic functions is replaced by the space of harmonic functions $f \colon V(\Gamma) \to \mathbf{C}$ obeying a polynomial growth condition

$$|f(v)| = O(d_\Gamma(v, v_0)^k)$$

for any fixed exponent k (where $v_0 \in V(\Gamma)$ is any fixed vertex). We do not know how to adapt our proof to this more general setting. Kleiner's argument can also be generalized to arbitrary vertex-transitive graphs, as here.

(3) It is especially notable that our bound on the dimension is linear in d, while Kleiner's bound is exponential in d^2 (see [K, §3.5] or [ST, Theorem 7.1]). On the other hand a sharp bound for $\dim \mathcal{H}_L(\Gamma)$ is known *a posteriori* by the following reasoning. First, Trofimov's extension of Gromov's theorem [T] implies that Γ is equivalent up to finite-fibre maps to a Cayley graph $\mathrm{Cay}(G, S)$, where G is a suitable finitely generated

virtually nilpotent quotient of $\mathrm{Aut}(\Gamma)$, and such that the maps are $\mathrm{Aut}(\Gamma)$-equivariant graph homomorphisms. One may check then that $\mathcal{H}_L(\Gamma) \cong \mathcal{H}_L(\mathrm{Cay}(G, S))$. Moreover, if $N \leq G$ is a finite-index nilpotent subgroup then every Lipschitz harmonic function on G may be identified with an affine-linear function on N/N' by a result of Alexopoulos [A] (see also [MPTY]), so $\dim \mathcal{H}_L(G) - 1$ is just the torsion-free rank of N/N', which is at most d by the Bass–Guivarc'h formula. Thus $\dim \mathcal{H}_L(\Gamma) \leq d + 1$.

This deep exact result notwithstanding, it is desirable to have a direct bound on $\dim \mathcal{H}_L(G)$ that is close to optimal. Shalom and Tao [ST] obtain a superpolynomial lower bound on the growth of non-virtually-nilpotent groups, by finding a quantitative "single-scale" analogue of Kleiner's proof of Gromov's theorem. The exponential dimension bound on $\dim \mathcal{H}_L(G)$ given by Kleiner's argument is an obstacle (though seemingly not the unique one) to improving the form of the superpolynomial rate they obtain.

2. An entropy bound

The key entropy inequality was used implicitly in [O, §3–4], but a particularly clear formulation is given in Benjamini–Duminil-Copin–Kozma–Yadin [BDCKY, Appendix A]. To introduce it we briefly recall some information-theoretic formalism. The *entropy* of a finitely supported probability measure $p\colon \Omega \to [0, 1]$ (identified with its mass function) is defined by

$$H(p) = \sum_{x \in \Omega} p(x) \log(1/p(x)).$$

We will also write $H[X]$ for $H(p)$ if X is a finitely supported random variable on Ω with law p (we use square brackets to indicate that a function is applied to the distribution of a random variable rather than its value). For a jointly distributed random variable (X, Y) on say $\Omega = \Omega_1 \times \Omega_2$, we write $H[X, Y]$ to mean $H[(X, Y)]$ etc., and define the *conditional entropy*

$$H[X \mid Y] = \sum_{y \in \Omega_2} p_Y(y) H[X \mid Y = y] = H[X, Y] - H[Y]$$

where $p_Y : \Omega_2 \to [0, 1]$ is the marginal density of Y. The *mutual information* of coupled random variables X and Y is

$$I[X; Y] = H[X] + H[Y] - H[X, Y].$$

The *entropy subadditivity inequality* states $H[X, Y] \leq H[X] + H[Y]$; equivalently, $I[X; Y] \geq 0$. See e.g. Gray [G1] for further background.

Lemma 2.1. *Let (X, Y) be a finitely supported coupled pair of random variables on some sample space $\Omega_1 \times \Omega_2$ and let $f\colon \Omega_1 \to \mathbf{C}$ be a complex-valued function. Then*

$$\mathbf{E}_Y \left[\left| \mathbf{E}_{X|Y}[f(X) \mid Y] - \mathbf{E}_X[f(X)] \right| \right] \leq 2 I[X; Y]^{1/2} \left(\mathbf{E}_X \left[|f(X)|^2 \right] \right)^{1/2}.$$

Proof. See [BDCKY, Corollary 18]. □

Now consider a connected, locally finite, vertex-transitive graph Γ as above. For $v \in V(\Gamma)$, let

$$v = X_0^v,\ X_1^v,\ X_2^v,\ \dots \tag{2.1}$$

be the simple random walk on Γ based at v, where X_n^v is chosen uniformly at random from the neighbourhood $N_\Gamma(X_{n-1}^v)$ of X_{n-1}^v for each $n \geq 1$, independently of earlier choices. Let μ_n^v be the law of X_n^v.

Since Γ is vertex-transitive, the measures μ_n^v and μ_n^w are the same up to an automorphism of Γ. In particular $H(\mu_n^v)$ does not depend on v, so we may define $H_n := H(\mu_n^v)$. It follows directly that $H\big[X_{n+k}^v \mid X_k^v\big] = H_n$ for any $n, k \geq 0$ and $v \in V(\Gamma)$, as $(X_{n+k}^v \mid X_k^v = w)$ has law μ_n^w. The *incremental entropy* of the random walk is $h(n) = H_n - H_{n-1}$. Note that

$$I[X_n^v; X_1^v] = H[X_n^v] - H[X_n^v \mid X_1^v] = H_n - H_{n-1} = h(n). \tag{2.2}$$

Lemma 2.2. *Let $f : V(\Gamma) \to \mathbf{C}$ be a complex-valued function, let $v \in V(\Gamma)$, and let $Y = X_1^v$. Then*

$$\mathbf{E}_Y\big[\big|\mathbf{E}[f(X_n^v) \mid Y] - \mathbf{E}[f(X_n^v)]\big|\big] \leq 2h(n)^{1/2}\big(\mathbf{E}\big[|f(X_n^v)|^2\big]\big)^{1/2}.$$

Proof. This is immediate from Lemma 2.1 and (2.2) (cf. [BDCKY, Proposition 19]). □

One can show that $h(n)$ is nonnegative, monotonically decreasing, hence convergent, and

$$\lim_{n\to\infty} h(n) = \lim_{n\to\infty} (H_n/n) \leq \lim_{n\to\infty} (\log b(n)/n). \tag{2.3}$$

Similarly,

$$\liminf_{n\to\infty} nh(n) \leq \liminf_{n\to\infty} \big(H_n/\log n\big) \leq \liminf_{n\to\infty} \big(\log b(n)/\log n\big). \tag{2.4}$$

Indeed, if $h(n) \geq c/n$ for some constant $c > 0$ and all sufficiently large n then

$$H_n = \sum_{m=1}^{n} h(m) \geq c \log n - O(1)$$

for all n, which implies $\liminf(H_n/\log n) \geq c$. The second inequality of (2.4) holds exactly as in (2.3), namely because $H_n \leq \log b(n)$ (since X_n^v is supported on $B_\Gamma(v, n)$).

Remark 2.3. There are examples such as the Cayley graph of the lamplighter group $G = C_2 \wr \mathbf{Z}$ for which $\lim h(n) = 0$ while $\lim \log b(n)/n > 0$. Thus the inequality (2.3) is not sharp, even qualitatively. On the other hand, regarding (2.4), $\liminf nh(n) < \infty$ if and only if $\liminf(\log b(n)/\log n) < \infty$. Indeed, if $nh(n) \leq C$ then $H_{2n} - H_n = \sum_{m=n+1}^{2n} h(m) \leq nh(n) \leq C$. If this holds for infinitely many n and a fixed constant C then Γ has polynomial growth by [B, Theorem 1.3]. This answers a side question in [O].

3. Lipschitz harmonic functions

We now turn to the proof of Theorem 1.1. In view of (2.4), it suffices to prove the claimed bound

$$\dim \mathcal{H}_L(\Gamma) \le 4r^2 d + 1$$

with $d = \liminf_{n\to\infty} nh(n)$. Let $V = \mathcal{H}_L(\Gamma)$ and $W = V/\mathbf{C}$, where $\mathbf{C} \subseteq V$ denotes the subspace of constant functions. Then $\|\cdot\|_{\mathrm{Lip}}$ defines a seminorm on V and a genuine norm on W. The idea of the proof of Theorem 1.1 is that $\|\cdot\|_{\mathrm{Lip}}$ is actually equivalent to a natural inner product norm, which implies that W is finite-dimensional.

For $v \in V(\Gamma)$ write $v = X_0^v, X_1^v, X_2^v, \dots$ for the random walk on Γ starting from v as in (2.1). For $f : V(\Gamma) \to \mathbf{C}$, define positive semi-definite hermitian quadratic forms

$$\begin{aligned} E^v(f) &= \mathbf{E}\big[|f(X_1^v) - f(X_0^v)|^2\big], \\ I_n^v(f) &= \mathbf{E}\big[|f(X_{n+1}^v) - f(X_n^v)|^2\big], \\ J_n^v(f) &= \tfrac{1}{n}\mathbf{E}\big[|f(X_n^v) - f(X_0^v)|^2\big]. \end{aligned}$$

These polarize to binary hermitian forms in the usual way, and we use the notation $E^v(f_1, f_2)$ etc. for the binary forms. Note that $I_n^v(f) \le \|f\|_{\mathrm{Lip}}^2$ for any $f \in V$, since $d_\Gamma(X_{n+1}^v, X_n^v) \le 1$.

If f is harmonic, we have that (for any $v \in V(\Gamma)$)

$$\mathbf{E}_{X_{n+1}^v | X_n^v}\big[f(X_{n+1}^v) \mid X_n^v\big] = \frac{1}{r}\sum_{y \in N_\Gamma(X_n^v)} f(y) = f(X_n^v);$$

that is, $f(X_n^v)$ is a martingale. It follows that the differences $f(X_1^v) - f(X_0^v), f(X_2^v) - f(X_1^v), \dots, f(X_n^v) - f(X_{n-1}^v)$ are uncorrelated, and so the variance of their sum is the sum of their variances:

$$J_n^v(f) = \frac{1}{n}\sum_{m=0}^{n-1} I_m^v(f). \tag{3.1}$$

In particular, $J_n^v(f) \le \|f\|_{\mathrm{Lip}}^2$ for all $f \in V$.

Let $(n_i)_{i\ge1}$ be a strictly increasing sequence such that $n_i h(n_i) \to d$ as $i \to \infty$. Pick a nonprincipal ultrafilter p containing $\{n_i : i \ge 1\}$ and define $J^v(f) = \lim_{n\to p} J_n^v(f)$ for $f \in V$. Note that J^v is again a hermitian quadratic form.

Lemma 3.1. *Assume $\lim h(n) = 0$ (e.g., $d < \infty$). Then $J = J^v$ does not depend on v.*

Proof. Define $F\colon V(\Gamma) \to \mathbf{R}$ by $F(v) = E^v(f)$. Then $\|F\|_\infty \le \|f\|_{\mathrm{Lip}}^2$ and $I_n^v(f) = \mathbf{E}\big[F(X_n^v)\big]$. Applying Lemma 2.2 to F, we get

$$\mathbf{E}_Y\left[\left|I_{n-1}^Y(f) - I_n^v(f)\right|\right] \le 2h(n)^{1/2}\,\|f\|_{\mathrm{Lip}}^2.$$

Letting $n \to \infty$, it follows that $|I_{n-1}^w(f) - I_n^v(f)| \to 0$ for all $vw \in E(\Gamma)$. By (3.1) it follows that $J_n^v(f) - J_n^w(f) \to 0$ as $n \to \infty$. Thus $J^v(f) = J^w(f)$ for all $vw \in E(\Gamma)$, and hence for all $v, w \in V(\Gamma)$ since Γ is connected. $\square$

Proposition 3.2. *For $f \in V$ we have*

$$J(f) \leq \|f\|_{\mathrm{Lip}}^2 \leq 4r^2 d \cdot J(f).$$

In particular J is positive-definite on W if $d < \infty$.

Proof. The first inequality is immediate from (3.1), so we focus on the second. By the martingale property, $f(v) = f(X_0^v) = \mathbf{E}_{X_n^v}[f(X_n^v)]$ and $f(X_1^v) = \mathbf{E}_{X_n^v | X_1^v}[f(X_n^v) \mid X_1^v]$. Applying Lemma 2.2 to $f - f(v)$, it follows that

$$\mathbf{E}\big[|f(X_1^v) - f(v)|\big] \leq 2h(n)^{1/2}\big(\mathbf{E}\big[|f(X_n^v) - f(v)|^2\big]\big)^{1/2} = 2\left(nh(n)J_n^v(f)\right)^{1/2}.$$

Letting $n \to p$, we get

$$\mathbf{E}\big[|f(X_1^v) - f(v)|\big] \leq 2d^{1/2}J(f)^{1/2}$$

and hence

$$\max_{w \in N_\Gamma(v)} |f(w) - f(v)| \leq 2rd^{1/2}J(f)^{1/2}$$

by definition of X_1^v. Taking the supremum over $v \in V(\Gamma)$ gives the result. $\square$

We can now complete the proof of Theorem 1.1. Fixing an edge $e = vw \in E(\Gamma)$, consider the linear functional $\varphi : W \to \mathbf{C}$ defined by $\varphi(f) = f(w) - f(v)$. By Proposition 3.2,

$$|\varphi(f)| \leq \|f\|_{\mathrm{Lip}} \leq C^{1/2}J(f)^{1/2},$$

where $C = 4r^2 d$. Hence, the dual norm of $\varphi \colon W \to \mathbf{C}$ obeys $\|\varphi\|_{J^*} \leq C^{1/2}$. Let $f_1, \ldots, f_k \in V$ be J-orthonormal. We have

$$\sum_{i=1}^k |\varphi(f_i)|^2 = \varphi\left(\sum_{i=1}^k \overline{\varphi(f_i)} f_i\right) \leq \|\varphi\|_{J^*} \left\|\sum_{i=1}^k \overline{\varphi(f_i)} f_i\right\|_J = \|\varphi\|_{J^*}\left(\sum_{i=1}^k |\varphi(f_i)|^2\right)^{1/2}$$

and so, rearranging,

$$\sum_{i=1}^k |f_i(w) - f_i(v)|^2 = \sum_{i=1}^k |\varphi(f_i)|^2 \leq \|\varphi\|_{J^*}^2 \leq C.$$

Fixing $v \in V(\Gamma)$ and averaging over $w \in N_\Gamma(v)$ gives

$$\sum_{i=1}^k E^v(f_i) \leq C, \qquad \text{and} \qquad \sum_{i=1}^k I_m^v(f_i) = \mathbf{E}\left[\sum_{i=1}^k E^{X_m^v}(f_i)\right] \leq C.$$

Now (3.1) implies

$$\sum_{i=1}^k J_n^v(f_i) = \sum_{i=1}^k \frac{1}{n} \sum_{m=0}^{n-1} I_m^v(f_i) \leq C.$$

It follows that

$$k = \sum_{i=1}^k J(f_i) = \lim_{n \to p} \sum_{i=1}^k J_n^v(f_i) \leq C.$$

This proves that $\dim W \leq C$, as claimed.

References

[A] G. K. Alexopoulos, *Random walks on discrete groups of polynomial volume growth*, Ann. Probab. **30** (2002), no. 2, 723–801. MR1905856

[ABGK] G. Amir, G. Blachar, M. Gerasimova, and G. Kozma, *Lipschitz harmonic functions on vertex-transitive graphs*, Electron. Commun. Probab. **29** (2024), Paper No. 21, 4. MR4756365

[B] G. Blachar, *Small entropy doubling for random walks and polynomial growth*, 2026.

[BDCKY] I. Benjamini, H. Duminil-Copin, G. Kozma, and A. Yadin, *Minimal growth harmonic functions on lamplighter groups*, New York J. Math. **23** (2017), 833–858. MR3690233

[G1] R. M. Gray, *Entropy and information theory*, Springer-Verlag, New York, 1990. MR1070359

[G2] M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73. MR623534

[K] B. Kleiner, *A new proof of Gromov's theorem on groups of polynomial growth*, J. Amer. Math. Soc. **23** (2010), no. 3, 815–829. MR2629989

[LP] J. R. Lee and Y. Peres, *Harmonic maps on amenable groups and a diffusive lower bound for random walks*, Ann. Probab. **41** (2013), no. 5, 3392–3419. MR3127886

[MPTY] T. Meyerovitch, I. Perl, M. Tointon, and A. Yadin, *Polynomials and harmonic functions on discrete groups*, Trans. Amer. Math. Soc. **369** (2017), no. 3, 2205–2229. MR3581232

[O] N. Ozawa, *A functional analysis proof of Gromov's polynomial growth theorem*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 3, 549–556. MR3831031

[ST] Y. Shalom and T. Tao, *A finitary version of Gromov's polynomial growth theorem*, Geom. Funct. Anal. **20** (2010), no. 6, 1502–1547. MR2739001

[T] V. I. Trofimov, *Graphs with polynomial growth*, Mat. Sb. (N.S.) **123(165)** (1984), no. 3, 407–421. MR735714

Sean Eberhard, Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, UK

Email address: sean.eberhard@warwick.ac.uk

Freddie Manners, UCSD Department of Mathematics, 9500 Gilman Drive #0112, La Jolla CA 92093, USA

Email address: fmanners@ucsd.edu